Resonant cancellation of off-resonant effects in a multilevel qubit

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Off-resonant effects are a significant source of error in quantum computation. This paper presents a group theoretic proof that off-resonant transitions to the higher levels of a multilevel qubit can be completely prevented in principle. This result can be generalized to prevent unwanted transitions due to qubit-qubit interactions. A simple scheme exploiting dynamic pulse control techniques is presented that can cancel transitions to higher states to arbitrary accuracy.

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Successful quantum computation depends on the accurate manipulation of the quantum states of the qubits [1]. In practice, qubits are subject to many sources of quantum errors including thermal fluctuations of the environment [2], qubit-qubit interactions [3], and intrinsic redundant degrees of freedom within a qubit such as the quasiparticle conduction in the superconducting qubits [4,5], and the effect of the higher levels in many practical qubit designs [5,6]. This paper proposes a dynamic pulse control technique that efficiently eliminates unwanted off-resonance transitions.

Various schemes to protect the qubit from qubit errors have been proposed that can be divided into two categories. The first one is the quantum error correction technique [7–12] where the qubit state is encoded by redundant qubits. Different errors project the qubit-extra-qubit system into different subspaces that can be determined by measuring the state of the extra qubits. By applying a transformation according to the measurement, the correct qubit state can be restored. This approach relies on large numbers of extra qubits to keep the errors from propagating. The second approach exploits 'bang-bang' control techniques [13] where the dynamics of the qubit and the environment is manipulated by fast pulses that flip the qubit. With the influence of the environment averaged out, the qubit evolves in the error-free subspace. This method relies on the ability to apply pulses rapidly compared with the correlation time of the environment. This is an open loop control method.

A particularly important form of intrinsic qubit error comes from the off-resonant transitions to the higher levels of a qubit when the qubit is being operated. Real qubits are not $S = 1/2$ spins that are perfect two level systems; redundant levels always exist that affect the information content of the qubit. The additional interaction that is introduced to achieve qubit operation by coupling the lowest two states of the qubit almost always includes unwanted couplings between the lowest two states and the higher levels. When the interaction is applied with frequency $\omega = \omega_2 - \omega_1$, resonant transition occurs between the lowest two states; meanwhile off-resonant transitions to the higher states are also switched on. These transitions deviate the phase and amplitude of the qubit state from perfect Rabi oscillation. Numerical simulations on the superconducting persistent current qubit (pc-qubit) [4,14] show that this deviation can be severe when the unwanted couplings are of the same order as the Rabi frequency.

In this letter, we study the effect of the higher levels on qubit dynamics during qubit operation by a group theoretic approach. We prove that the errors can be completely avoided by applying a time varying operation Hamiltonian. Then we generalize this result to the qubit-qubit interaction problem which can be mapped exactly onto the first one. Using the idea of dynamic pulse control [13], we design a pulse sequence that cancels the leakage to the higher levels to arbitrary accuracy with $O(N)$ number of pulses, N being the number of higher levels.

The leakage to higher levels has two significant characteristics. First, unlike the environmental fluctuations that affect the qubit only slightly (less than 10^{-4}) within one operation, the leakage changes the qubit dynamics on a time scale $1/\omega_0$ that is much shorter than the qubit operation time (about $1/\omega_{Rabi}$). Conventional quantum error correction corrects errors that occur with small probability and is not a suitable strategy to cancel the fast off-resonant transitions. Neither can we use the bang-bang method to average out [15] the leakage simply by manipulating the lowest two states, as the pulse induces these unwanted transitions at the same time. Second, ignoring all interactions with external variables, the leakage is coherent, although the coherent oscillation will collapse since the revival time is too long to be observed due to the large number of transitions of different frequencies [16]. As will now be shown, the coherent nature of the leakage implies that this type of error can be corrected by applying a control sequence that coherently modifies the qubit dynamics.

Consider a N level quantum system with the Hamiltonian $\mathcal{H}_0$, the lowest two states of which are chosen as the qubit states $|\uparrow\rangle$ and $|\downarrow\rangle$. The unitary transformations on this N dimensional Hilbert space form the N^2 dimensional compact Lie group U(N). Without other in-

teraction, the trajectory of the qubit follows the Abelian subgroup $\{e^{-i\mathcal{H}_0 t}, t \in R\}$.

Now apply to the qubit the perturbation $\mathcal{H}_I$, $[\mathcal{H}_0, \mathcal{H}_I] \neq 0$, to induce a desired transformation of the qubit. In most physical systems, unwanted transitions to the higher levels are simultaneously induced. For example, in the pc-qubit [4] operation, $(\mathcal{H}_\mathcal{I})_{mn} = 2\pi\delta f \langle m| \sin(2\phi_m + 2\pi f)|n\rangle \cos\omega t$, when the bias flux is modulated with rf components of amplitude δf and frequency ω. This perturbation has couplings between all the levels. By successive commutation of $\mathcal{H}_0$, $\mathcal{H}_I$, and their commutators until no independent operator appears, a Lie algebra $\mathcal{A}_I$ is created. In almost all cases, $\mathcal{A}_\mathrm{I} = \mathrm{u}(N)$ [17], $\mathrm{u}(N)$ being the Lie algebra of $\mathrm{U}(N)$. The only exception occurs in a zero measure subspace of $\mathrm{u}(N)$ when $\mathcal{H}_I$ and $\mathcal{H}_0$ are both in the same subalgebra of $\mathrm{u}(N)$. Thus, with almost all perturbations, the evolution operator can be any element in $\mathrm{U}(N)$; and transitions to higher levels are unavoidable with an initial state that only occupies the lowest two levels.

To prevent the transitions to the higher states at time t means to restrict the evolution operator $\mathcal{U}(t)$ to the submanifold of $\mathrm{U}(2) \oplus \mathrm{U}(N-2)$, $\mathrm{U}(2)$ being the unitary group on $\{|\uparrow\rangle, |\downarrow\rangle\}$ and $\mathrm{U}(N-2)$ on the remaining $N-2$ states. This applies $4(N-2)$ real domain restrictions on $\mathcal{U}(t)$: $\mathcal{U}(t)_{1k}, \mathcal{U}(t)_{2k} = 0, k = 3 \ldots N$. In contrast to a perfect qubit operation during which $\mathcal{U}(t)$ remains in the subspace $\mathrm{U}(2) \oplus \mathrm{U}(N-2)$ all the time, the qubit is allowed to stray away from this subspace if only it goes back to this subspace at the designated time t. The qubit dynamics can be manipulated by varying the strength and phase of the perturbation with time. As the N^2 dimensional Lie group $\mathrm{U}(N)$ is compact, any transformation can be reached at time t by adjusting the N^2+1 parameters in the following process [17]:

$$\mathcal{U}(t) = e^{-i\alpha\mathcal{H}_I t_{N^2}} e^{-i\alpha\mathcal{H}_0 t_{N^2-1}} \cdots e^{-i\alpha\mathcal{H}_0 t_1} \qquad (1)$$

where α is introduced to ensure that $t = \sum t_i$. By playing with the N^2+1 real parameters, the $4(N-2)$ real numbers in $\mathcal{U}(t)_{1k}, \mathcal{U}(t)_{2k}$ can be set to zero so that the state of the qubit stays in the $\{\uparrow, \downarrow\}$ space without leakage. Hence by turning the perturbation on and off $O(N^2)$ times, the lowest two states are completely decoupled from the higher states. $O(N^2)$ pulses give a sufficient condition that is required to achieve arbitrary transformation. As will be shown later in this letter, with proper arrangement, we can design a pulse sequence of $O(N)$ pulses to cancel the transitions to the higher levels. Unlike that in the quantum Zeno effect [18] where measurement is used to prevent the system from evolving, the dynamics in this process is described completely by unitary evolutions.

As the unwanted transitions are off-resonant transitions whose amplitudes decrease roughly as γ_{ij}/ω_i (ω_i is the energy of the ith level, γ_{ij} the coupling between level i and j), the influence of the levels with $\omega_i/\omega_0 \gg 1$ can be ignored. In the pc-qubit [4] the energies of the lower levels increase fast enough ($\omega_{10} > 10\omega_0$) that levels beyond $|10\rangle$ can be ignored. The energy of the ith level of the charge state qubit [5] increases as i^2; less levels affect the qubit dynamics than that in the pc-qubit. Hence the number N of the higher states that are involved in the qubit dynamics in real designs can be reasonably small. As a result, the number of pulses in the previous analysis is also reasonable.

One question to ask is whether there is any fundamental difference between the errors due to transitions to the higher levels and those due to the fluctuations of the environmental variables. Putting it in another way: what is the difference between the intra-qubit coupling in a multilevel qubit and the qubit-external-system coupling? In the following we will show that the N-level qubit can be mapped into interacting subsystems, and vice versa.

Let the initial state of a N-level qubit be $|\Psi_0\rangle = \alpha_1^{(0)}|1\rangle + \alpha_2^{(0)}|2\rangle$, $|1\rangle$ and $|2\rangle$ being the lowest two states. To map the qubit into two subsystems, we divide the N states into two subspaces SP_1 and SP_2 by adding the vacuum states $|V_1\rangle$ and $|V_2\rangle$ to the respective subspaces as $SP_1 = \{|V_1\rangle, |1\rangle, |2\rangle\}$ and $SP_2 = \{|V_2\rangle, |3\rangle, \ldots, |N\rangle\}$. Now the N dimensional Hilbert space of the original qubit is embedded in the $3(N-1)$ dimensional direct product space $SP_1 \otimes SP_2$. The states in the expanded space are $|\overline{\Psi}\rangle = \sum_{i,j} \beta_{i,j} |b_i^{(1)}\rangle |b_j^{(2)}\rangle$, where $b_i^{(1)}$ and $b_j^{(2)}$ are basis of the two subspaces respectively. The initial state is $|\overline{\Psi}_0\rangle = (\alpha_1^{(0)}|1\rangle + \alpha_2^{(0)}|2\rangle)|V_2\rangle$ in the expanded form. The unitary transformations in this expanded space forms the group $\mathrm{U}(3(N-1))$.

Perturbation introduces coupling between different states. When mapped to the expanded space, the perturbation $\overline{\mathcal{H}_I}$ connects states in the N dimensional subspace spanned by $\{|1\rangle|V_2\rangle, |2\rangle|V_2\rangle, |V_1\rangle|3\rangle, \ldots, |V_1\rangle|N\rangle\}$. So $\overline{\mathcal{H}_I}$ and $\overline{\mathcal{H}_0}$ create N^2 dimensional subalgebra $\mathrm{u}(N)$ in the expanded space. Under the perturbation, the wave function in the expanded space can be described as $|\overline{\Psi}\rangle = (\alpha_1|1\rangle + \alpha_2|2\rangle)|V_2\rangle + \sum_{i=3}^{N} \alpha_i |V_1\rangle|i\rangle$, where α_i are time dependent parameters evolving with the perturbation.

From this analysis, the higher levels in the qubit form an effective environment that interferes **strongly** with the lowest two levels. Interaction strength is the major difference between this effective environment and a real one [4]. The couplings between SP_1 and SP_2 are strong and comparable to the Rabi coupling that realizes qubit operation. In contrast, the interactions between the environmental oscillators and the qubit are weak due to the $O(1/\sqrt{V})$ factor that originates from the normalization of the extented modes [2]. So the thermal fluctuations are not enslaved to the qubit dynamics and can be treated classically. The strong interaction with the higher levels also explains why the error due to leakage occurs at such a short time that a particular strategy is required to correct the error. Another thing to mention is that this effective environment only comes with qubit operation, while the real environment affects the qubit all the time.

Hence we worry about the leakage only during qubit operation and choose to correct the leakage by controlling the operation process.

By reversing the mapping, interacting qubits can be modeled as one multilevel quantum system. One example is two interacting qubits with basis $|b_i^{(1)}\rangle, i = 1 \ldots N_1$, and $|b_j^{(2)}\rangle, j = 1 \ldots N_2$, respectively. Labeling the state $|b_i^{(1)}\rangle|b_j^{(2)}\rangle$ as $|(i-1)N_2 + j\rangle$, we have the N_1N_2 basis for the equivalent multilevel qubit. In the same way, n two-level qubits form a quantum system of 2^n levels. Here the number of states grows exponentially with the number of qubits as the entanglement between qubits has to be included in a single system now [19]. Perturbation applied to one of the qubits can cause unwanted couplings within the 2^n levels, and induce off-resonant transitions that affect the qubit performance. Similar to the couplings of the multilevel qubit, these couplings are also strong and cause fast errors. Taking the pc-qubit as an example, the interaction between the two qubits is [4]: $\mathcal{H}_2 = J_z\sigma_z^{(1)}\sigma_z^{(2)} + J_x(\sigma_z^{(1)}\sigma_x^{(2)} + \sigma_x^{(1)}\sigma_z^{(2)})$, where J_z and J_x terms are due to the inductance coupling between qubit circuits. When a $\sigma_x^{(1)}$ term is applied to rotate the first qubit, the second qubit will be involved and qubit dynamics will be changed.

Although the mapping of the multilevel qubit and the multiqubit system into each other is just another way of looking at the same problem, it shows that the errors due to the qubit-qubit interactions [3] can be treated by the same approach as is used in cancelling the interference of the higher levels. Again we turn to the idea of coherent pulse control that is exploited in the higher state problem. Now the number of pulses increases exponentially with the number of interacting qubits, but it doesn't cause a disaster in real designs where only the nearest neighbour qubit interactions are important and n can be made small in the qubit layout geometry.

To illustrate the general idea of exploiting dynamic pulse control to cancel the errors due to the higher states, we give an example of pulse sequence that completely cancels the transitions to the higher levels with $O(N)$ pulses. Let us start from a three level system with eigenvalues $\omega_i, i = 1, 2, 3$. The energy difference between level i and j is shorthanded as ω_{ij}. An interaction $\mathcal{H}_I$ that couples level i and j by γ_{ij} is applied to operate the qubit. When the third level is not present, γ_{12} is the Rabi frequency of the lowest two states. For simplicity, we ignore the diagonal couplings γ_{ii} as $\gamma_{ii} \ll \omega_i$. As will become clear, the effectiveness of the designed pulse sequence depends on the condition $|\gamma_{ij}/\omega_{ij}| \ll 1$ which is satisfied in most qubits.

The Hamiltonian in the interaction picture is $\mathcal{H}_{int} = e^{i\mathcal{H}_0 t}\mathcal{H}_I e^{-i\mathcal{H}_0 t}\cos(\omega t + \phi)$, ω being the pulse frequency. The wave function $\Psi(t) = [u\, v\, w]^T$ evolves according to the equation $i\frac{\partial\Psi(t)}{\partial t} = \mathcal{H}_{int}\Psi(t)$. When the perturbation is weak, this equation is integrated order by order as:

$$\begin{aligned}\Psi(t) &= \Psi(0) + \textstyle\int_0^t dt'\mathcal{H}_{int}(t')\Psi(0) \\ &+ \textstyle\int_0^t dt' \int_0^{t'} dt''\mathcal{H}_{int}(t')\mathcal{H}_{int}(t'')\Psi(0) + \cdots ,\end{aligned} \tag{2}$$

The cosine function is used in the rf pulse instead of the single frequency rotating wave. In many systems, no physical correspondence of the rotating frame exists. For example, the circuit of the pc-qubit is biased by z direction magnetic flux and the perturbation is high frequency modulation of the z flux. No rotating frame of transverse field can be defined for the oscillating flux.

Our strategy to reduce the unwanted transitions is to divide the qubit operation into short intervals of t_0 and attach additional pulses to each operation pulse to correct errors from this short interval. The operation pulse is in resonant with ω_{21} of the lowest two states. Besides rotating the qubit between the level 1 and 2, it brings up off-resonant transitions between the third level and these two levels through the couplings γ_{13} and γ_{23}. Then the same perturbation is applied in two other pulses with different frequencies, amplitudes and phases as: $\alpha_{31}\mathcal{H}_I\cos(\omega_{31}t + \phi_{31})$ and $\alpha_{32}\mathcal{H}_I\cos(\omega_{32}t + \phi_{32})$, both for time t_0, to cancel the unwanted transitions to the third level. This three-piece sequence is repeated τ_{op}/t_0 times to finish the qubit operation. The time t_0 satisfies $1/\omega \ll t_0 \ll 1/\gamma_{ij}, i, j = 1, 2, 3$ with both $1/\omega_{21}t_0$ and $\gamma_{ij}t_0$ being small parameters of the same order. Thus we have two small parameters in this process. This is crucial for this simple pulse sequence to work.

Starting with an initial wave function $\Psi(0) = [u_0\, v_0\, w_0]^T$, $w_0 = 0$, after the ω_{21} pulse, the third level has the component:

$$\begin{aligned}w = &\ u_0\left(\frac{\gamma_{13}^*(e^{-i(\omega_{21}-\omega_{31})t_0}-1)}{\omega_{21}-\omega_{31}} - \frac{\gamma_{13}^*(e^{i(\omega_{21}+\omega_{31})t_0}-1)}{\omega_{21}+\omega_{31}}\right) \\ &+ v_0\left(\frac{\gamma_{23}^*(e^{-i(\omega_{21}-\omega_{32})t_0}-1)}{\omega_{21}-\omega_{32}} - \frac{\gamma_{23}^*(e^{i(\omega_{21}+\omega_{32})t_0}-1)}{\omega_{21}+\omega_{32}}\right) , \\ &+ u_0\theta_u + v_0\theta_v\end{aligned} \tag{3}$$

where θ_u and θ_v are of third order. The main components in w are second order terms that depend on the initial wave function u_0 and v_0 linearly. With t_0 satisfying $e^{2i\omega_{21}t_0} = 1$, u and v have third order deviations from the correct two-level rotation. The other two pulses are then applied to cancel the w component. The ω_{31} pulse induces a resonant transition between level one and level three to cancel the u_0 term in w; the ω_{32} pulse induces a resonant transition between level two and level three to cancel the v_0 term in w. The amplitudes and phase shifts of these two pulses can be expanded in ascending order as:

$$\begin{aligned}\alpha_{31}e^{i\phi_{31}} &= \alpha_{31}^{(1)}e^{i\phi_{31}^{(1)}} + \alpha_{31}^{(2)}e^{i\phi_{31}^{(2)}} + \cdots \\ \alpha_{32}e^{i\phi_{32}} &= \alpha_{32}^{(1)}e^{i\phi_{32}^{(1)}} + \alpha_{32}^{(2)}e^{i\phi_{32}^{(2)}} + \cdots\end{aligned} , \tag{4}$$

The first order coefficients cancel the second order terms in w and modify the higher order terms θ_u and θ_v when:

$$\begin{aligned}\alpha_{31}^{(1)}e^{i\phi_{31}^{(1)}} &= \frac{e^{-i(\omega_{21}-\omega_{31})t_0}-1}{i(\omega_{21}-\omega_{31})t_0} - \frac{e^{i(\omega_{21}+\omega_{31})t_0}-1}{i(\omega_{21}+\omega_{31})t_0} \\ \alpha_{32}^{(1)}e^{i\phi_{32}^{(1)}} &= \frac{e^{-i(\omega_{21}-\omega_{32})t_0}-1}{i(\omega_{21}-\omega_{32})t_0} - \frac{e^{i(\omega_{21}+\omega_{32})t_0}-1}{i(\omega_{21}+\omega_{32})t_0}\end{aligned} , \tag{5}$$

It turns out that the nth order terms of w after the two correction pulses include linear terms of $\alpha_{31}^{(n-1)}$ and $\alpha_{32}^{(n-1)}$, and complicated terms that depend on $\alpha_{3i}^{(k)} e^{i\phi_{3i}^{(k)}}$ $(k = 1 \ldots (n-2))$. So, for any n, $\alpha_{31}^{(n-1)}$ and $\alpha_{32}^{(n-1)}$ can be determined by the lower order components of α_{31} and α_{32} to cancel the nth order of w. As a result, the transitions to the third level can be completely erased. The parameters α_{31} and α_{32} do not depend on the initial wave function u_0 and v_0. This is similar to solving the wave function in the perturbation theory where the higher order terms are derived after the lower order ones.

After the kth pulse sequences, with $w = 0$, the wave function of u and v is:

$$\begin{bmatrix} u_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} \cos\bar{\varphi} + \bar{s}_u & -i\sin\bar{\varphi} + \bar{t}_u \\ -i\sin\bar{\varphi} + \bar{t}_v & \cos\bar{\varphi} + \bar{s}_v \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix}, \quad (6)$$

where $\bar{\varphi} = \gamma_{12} t_0$ is the phase rotation of the two-level qubit; the $\bar{s}$ and $\bar{t}$ terms are of third order. As $w = 0$, this is a unitary transformation that deviates from the original Rabi oscillation by third order corrections. The matrix can be written as $\mathcal{U}(t_0) = \exp\left(-i(\gamma_{12}\sigma_x + \delta_0 + \sum_i \delta_i \sigma_i) t_0\right)$, where δ_i are third order small numbers that can be determined by known parameters and do not depend on the index k. This is a renormalization of the qubit operation γ_{12} with the third level decoupled.

This correction strategy is easily generalized to $N (N \geq 3)$ level system. By applying rf pulses with frequencies $\omega_{i1}, \omega_{i2}, i = 3 \ldots N$, the transitions to the higher levels are completely erased. Assuming no particular symmetry between the states, $2(N-2)$ pulses are required in this process.

One may wonder why this simple pulse sequence works so well to correct the transitions to the higher states. For $N-2$ higher levels, to decouple these levels is to exert $4(N-2)$ real domain restrictions on the transformation matrix: $\mathcal{U}_{1i}, \mathcal{U}_{2i} = 0, i = 3 \ldots N$. Our tools are the Hamiltonians $\mathcal{H}_0$ and $\mathcal{H}_I$ that create the whole u(N) algebra by commutation. Our pulse sequence $\mathcal{U}(t_0) = \Pi_{i,\beta} P(\alpha_{i\beta}, \phi_{i\beta}) e^{-i\int \mathcal{H}_I \cos\omega_{21} t' dt'}$ ($i = 3 \ldots N$ and $\beta = \uparrow, \downarrow$), $P(\alpha_{i\beta}, \phi_{i\beta}) = e^{-i\int \mathcal{H}_I \alpha_{i\beta} \cos(\omega_{i\beta} t' + \phi_{i\beta}) dt'}$, contents $4(N-2)$ free parameters. By choosing proper pulse sequence, we can achieve the decoupling with proper pulse parameters.

In conclusion, we discussed the errors due to unwanted transitions to the higher states of a qubit during qubit operation. It was shown by a group theoretic argument that these errors can be completely prevented in principle. Then we generalized the result to the errors due to qubit interactions, which can also be prevented when the number of coupled qubits is not large. A simple pulse sequence that modifies the qubit dynamics and cancels off-resonant transitions to arbitrary accuracy with $O(N)$ pulses was proposed to illustrate the general analysis. Our results showed that the idea of dynamic pulse control [13] also works for the fast quantum errors due to the higher states of a qubit. These results suggest that dynamic pulse control, together with conventional quantum error correction, can function as a powerful tool for performing accurate quantum computation in the presence of errors.

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